Brian Blais' Homemade Guide to Special Relativity

This is a guide to describe the basic ideas of relativity. Currently it assumes some knowledge of relativity already but when I have more time, I will make the introduction more complete. Until then one can consult any introductory text, such as the excellent book *Spacetime Physics* by Taylor and Wheeler.

1 Introduction to Relativity

There is one principle of relativity, from which many consequences come:

> *Principle of Relativity:* All the laws of physics are the same in every inertial reference frame[1].

Some immediate consequences of this principle are

- the *form* of the laws of physics is the same in every inertial reference frame (IRF)
- numerical values of the physical constants are the same in every IRF
- *speed of light*, c, is the same in every IRF
- the *interval*, defined as $\Delta(ct)^2 - (\Delta x)^2 = \Delta(ct')^2 - (\Delta x')^2$, is measured to be the same in every IRF

It is then easy to derive the well-known Lorentz transformation equations, those equations which allow us to convert between events measured in one frame to those measured in the other. Two observers in different frames may not measure the same events at the same place or time, but they must agree on all of the laws of physics regarding those events in order for the principle of relativity to hold.

2 Derivation of the Lorentz Transformations

In all the examples we will assume a frame "M" and a frame "M'", where the M' frame is moving with a velocity v along the x direction relative to the frame M. Likewise, the M frame is moving with a velocity $-v$ relative to the M' frame. For example, we might be standing on the ground watching a train pass by us. Our measurements of time and length are said to be made in the M frame, while measurements of time and length on the train are said to be made in the M' frame. I will often refer to the frames M and M' as the unprimed and the primed frame, respectively. We will wear the cap of the observer in the unprimed frame, and then switch and wear the cap of the observer in the primed frame. Coordinates in the unprimed frame will be denoted with unprimed variables, (x,y,z,t), while the ones in the primed frame will be denoted with primed variables, (x',y',z',t'). For simplicity, in all the examples we will assume that the motion is in the x direction, so we can ignore y and z.

The Lorentz transformations are just $x(x',t',v)$ and $t(x',t',v)$, where v is the velocity of the primed frame relative to the unprimed frame, or $v = x/t$. If we look at the motion of the origin of the frame M', $x' = 0$, then we can simply derive the transformations. We start with a few definitions.

$$v = \frac{x}{t} \tag{2.1}$$

$$(ct)^2 - x^2 = (ct')^2 - x'^2 \tag{2.2}$$

[1] An *inertial reference frame* is used here to denote any system moving at a constant velocity, with no gravitational influences

from these, including the assumption $x' = 0$, we obtain

$$\begin{aligned}
\beta \equiv \frac{v}{c} &= \frac{x}{ct} \\
x &= \beta(ct) \\
(ct)^2 - x^2 &= (ct')^2 \\
&= (ct)^2 - \beta^2(ct)^2 \\
\Rightarrow \quad & (ct) = \frac{1}{\sqrt{1-\beta^2}}(ct') \equiv \gamma(ct') \\
& x = \beta(ct) = \beta\gamma(ct')
\end{aligned}$$

If we lift the assumption on x', then we get equations of the form.

$$\begin{aligned}
(ct) &= \gamma(ct') + Ax' \\
x &= \beta\gamma(ct') + Bx'
\end{aligned}$$

where the constants A and B are yet to be determined.

$$\begin{aligned}
(ct)^2 - x^2 &= (ct')^2 - x'^2 \\
&= (\gamma(ct') + Ax')^2 - (\beta\gamma(ct') + Bx')^2 \\
&= (ct')^2 + 2\gamma(A - \beta B)x'(ct') + (A^2 + B^2)x'^2 \\
\Rightarrow \quad & A - \beta B = 0 \\
& (A^2 + B^2) = 1
\end{aligned}$$

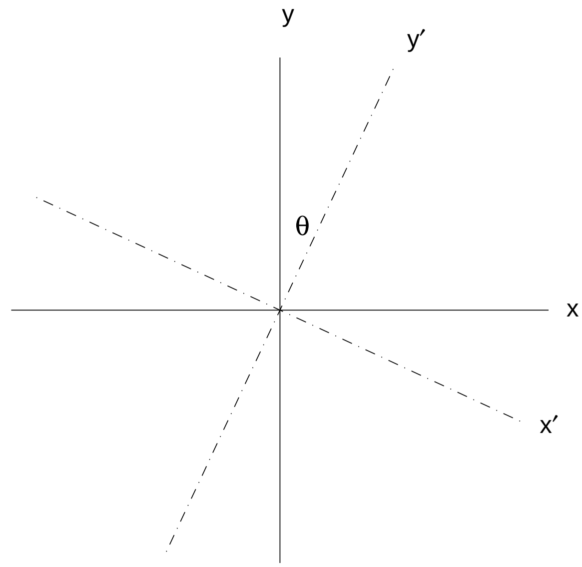
which gives us the final Lorentz transformation equations:

$$\boxed{\begin{aligned}
x &= \gamma x' + \beta\gamma(ct') && (2.3) \\
(ct) &= \gamma(ct') + \beta\gamma x' && (2.4) \\
x' &= \gamma x - \beta\gamma(ct) && (2.5) \\
(ct') &= \gamma(ct) - \beta\gamma x && (2.6)
\end{aligned}}$$

These equations give us all the relativistic effects, such as *length contraction* (ie. rulers in moving frames appear short) and *time dilation* (ie. clocks in moving frames appear to run slowly). They are used to translate between measurements in one frame to measurements in the other.

3 Spacetime Diagrams

We can help ourselves visualize the Lorentz transformations by drawing diagrams wherein we can draw spacetime events, and see how they look in the two frames. These diagrams are called Minkowski spacetime diagrams. We can draw a fairly good analogy from normal $x - y$ plots of rotations, so I will start with that. Let's say that we have two sets of axes, $x - y$ and $x' - y'$, rotated relative to each other by an angle θ.



Presuming that the observers using each set of axis are using the same length units then we need to come up with a consistent way to calibrate the axes. In normal Euclidean space, such as we have here, the *lengths* of lines are preserved. Or, in other words, the coordinates satisfy the equation

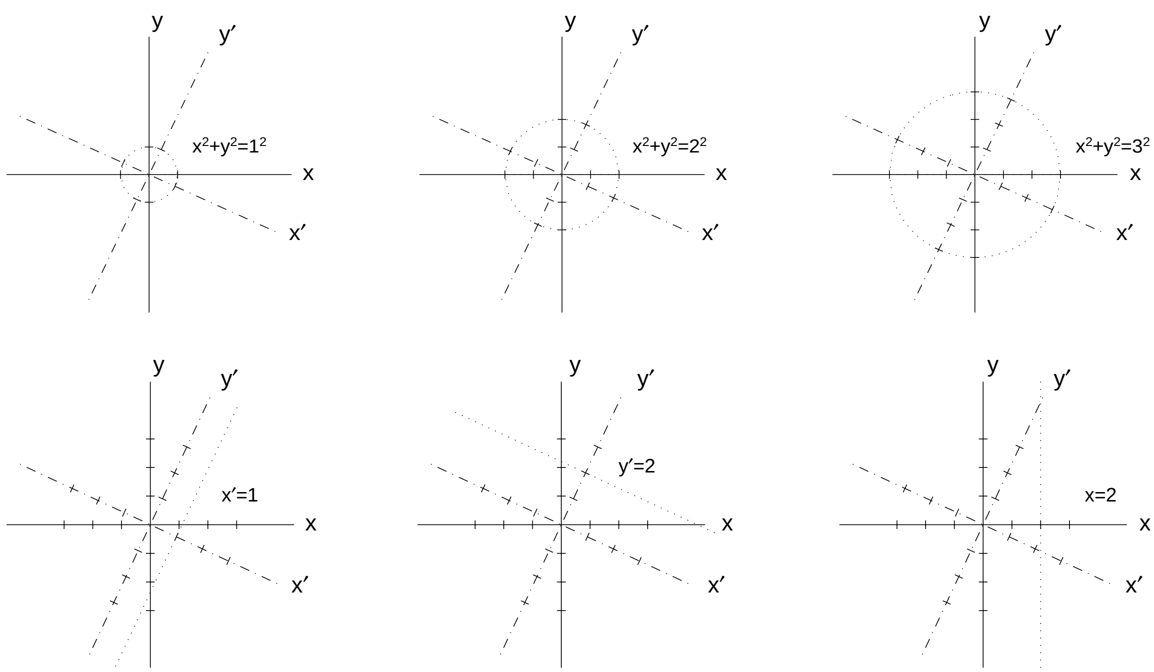
$$x^2 + y^2 = x'^2 + y'^2 \tag{3.7}$$

which is the equation of a circle. This equation is the analogue of the invariance of the interval equation in special relativity, namely

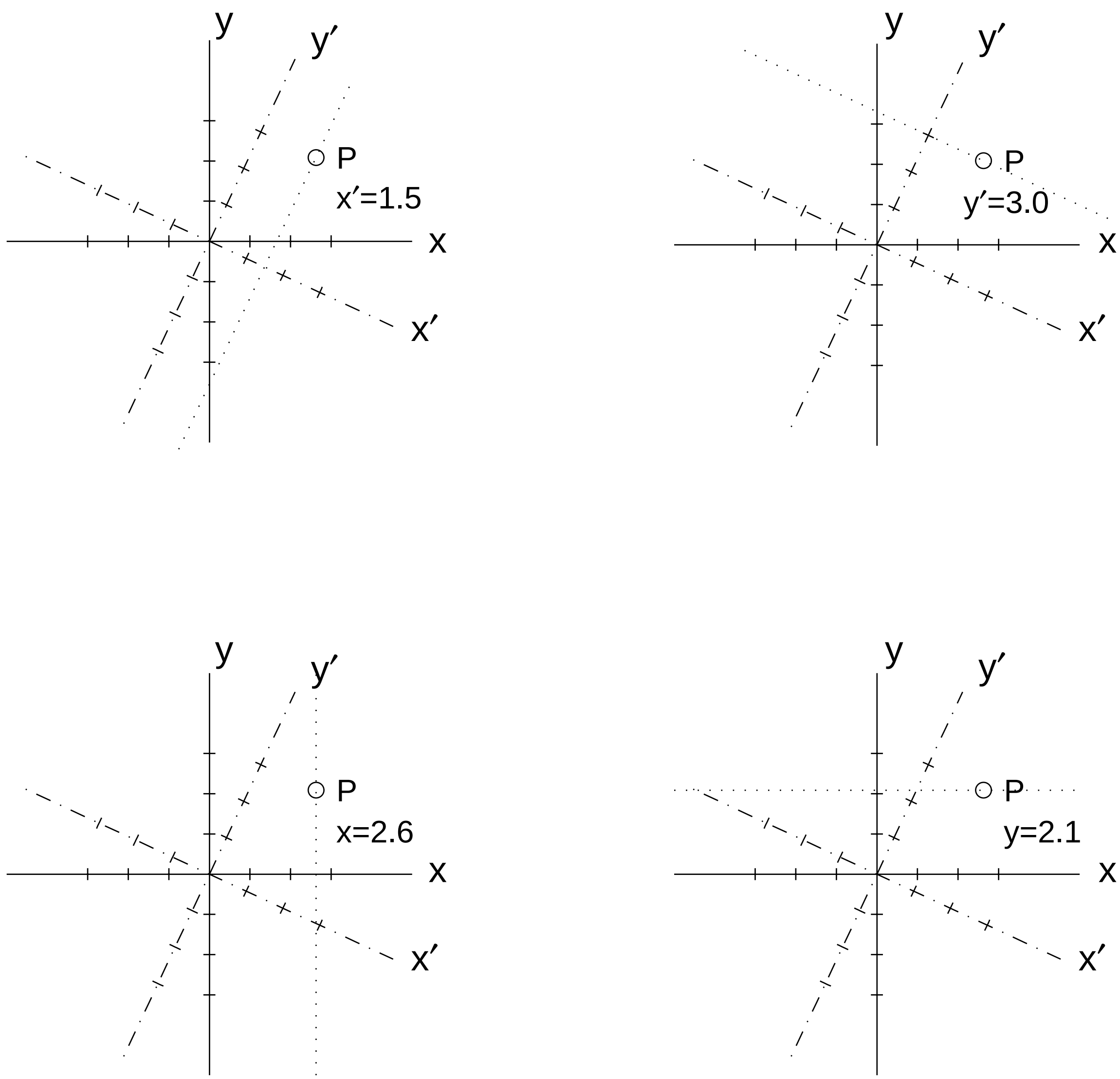
$$(ct)^2 - x^2 = (ct')^2 - x'^2 \tag{3.8}$$

which is the equation of a hyperbola. This will lead to all of the fairly non-intuitive aspects of SR.

Thus, calibration is done by drawing the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 2$, etc. and marking off where they strike the $x' - y'$ axes. Points with the $x' = 1$, for instance, all fall on a line parallel to the y' axis crossing the point were the $x^2 + y^2 = 1$ circle crosses the x' axis. The following pictures illustrate this.



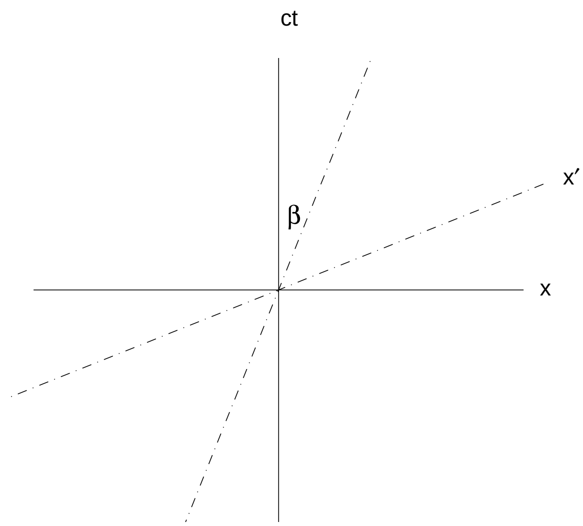
We can now determine the coordinates of any point P in either the $x-y$ frame or the $x'-y'$ frame, by drawing lines through the point P parallel to each axis.



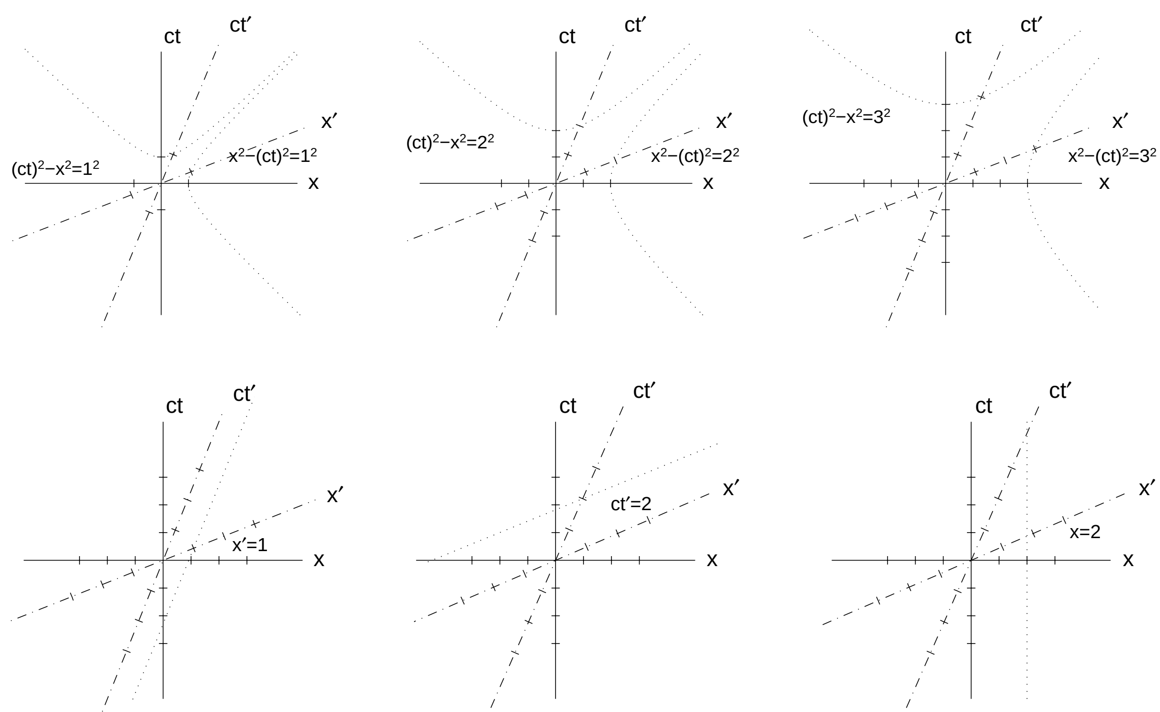
This might all seem obvious but when we do the same recipe for the spacetime diagrams we cannot rely on our Euclidean intuition. We now do the same steps as above, but with $x-(ct)$ axes:

- Draw the $x-(ct)$ and $x'-(ct')$ axes
- Calibrate with the invariant interval defined in Equation 3.8
- Look at events (ie. points P on the graph) in each of the frames using lines parallel to the axes

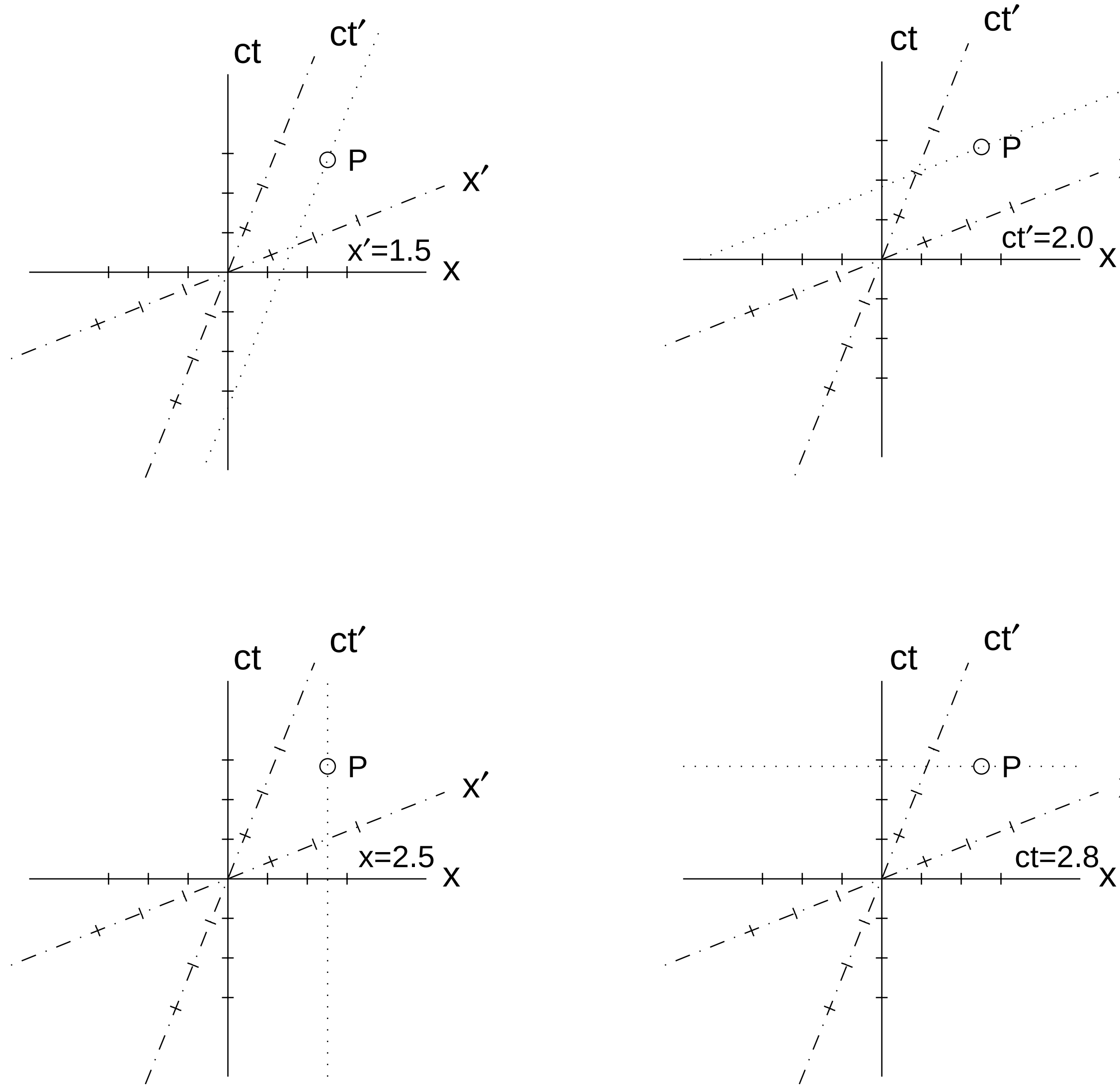
To draw the axes, we go back to the Lorentz transformations, Equations 2.3 and 2.4 and draw the $x'=0$ and $t'=0$ lines. Notice that the slope of the $x'=0$ line is $\beta \equiv v/c$, which approaches 1 as $v \to c$. Thus, an unmoving object would have the same axes as the unprimed frame, and a 45 degree line would imply motion at the speed of light.



Next we calibrate with the equal interval hyperboli, and look at lines of the same x' or ct' value, just as we did before.



Again, as before, we can figure out the coordinates of a point P in both of the coordinate systems by drawing lines parallel to the axes.



4 Example 1: Spacetime Diagram of a Baseball

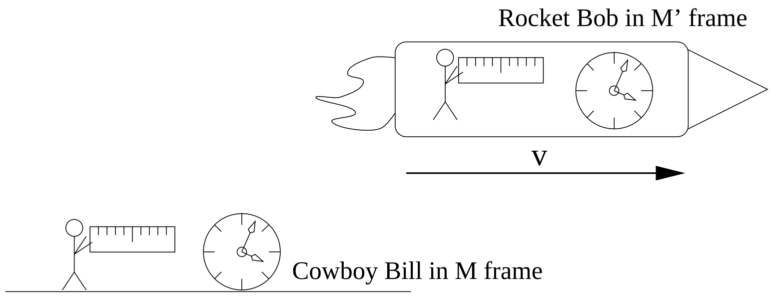
Before attacking the more standard, but also more abstract, examples of special relativity it is good to deal with an example involving a common object like a baseball. This will answer the question *why don't we see the effects of special relativity every day?*.

A baseball can move about 100 mph. What we need to do is compare events in the frame of the baseball, defined to be the M' frame, versus events in the frame of observers on the ground, defined to be the M frame. To do this we follow the same recipe outlined earlier to calibrate the axes for a moving object. Here, the slope of the baseball's time axis (t' axis) is $\beta = v/c \approx 100\ \text{mph}/c \approx 2 \cdot 10^{-7}$. One can clearly see that the "ground observer" axes and the "baseball" axes would be indistinguishable! They would, then, have the same calibration which would lead to identical measurements of time and space. Even if one were going 10 or 100 times as fast, the deviation from the non-moving frame would be essentially insignificant (though it could get measured using an atomic clock).

Therefore, the effects of special relativity will only be seen for objects moving at speeds of a significant fraction of the speed of light.

5 Example 2: A Mild Disagreement about Meter Sticks

In all of the following examples I will take the velocity of the primed frame to be $v = \frac{3}{5}c$ in the x direction, as in the following picture.

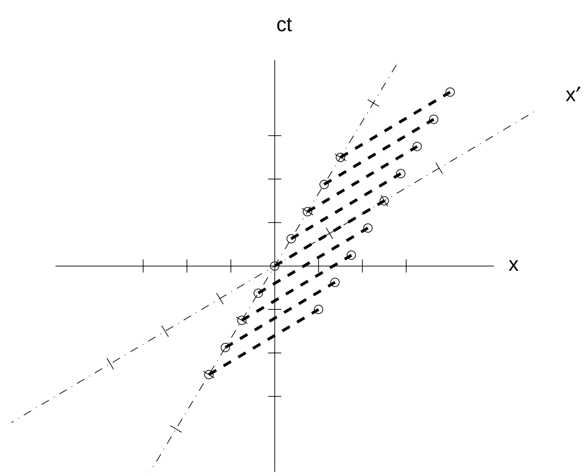


Considering the phenomenon of *length contraction*, Rocket Bob and Cowboy Bill appear to have a disagreement on the length of the other's ruler. Bill says that Bob's ruler is shorter, because Bob is the one moving. Bob says Bill's is shorter, because (to Bob) Bill is the one moving. They can't both be right...or can they? The fundamental idea here is that one cannot consider length contraction alone or the problem becomes contradictory. This is true for nearly all of the relativistic "paradoxes". The solution comes from considering events in space-time, and using the Lorentz transformation equations (or equally, the space-time diagrams), in order to take into account both length contraction and time dilation.

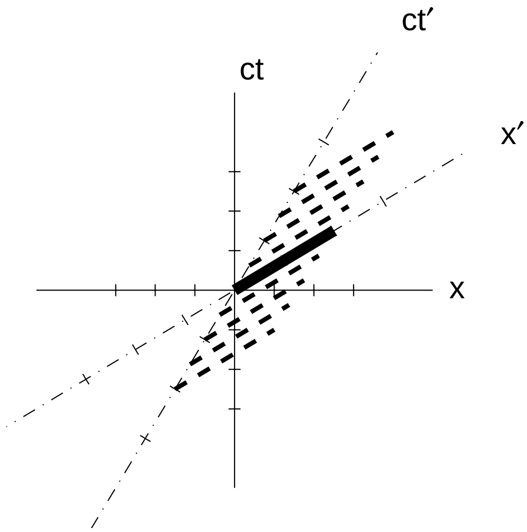
The Picture

As Rocket Bob moves past Cowboy Bill we can map out all of the (x, t) and (x', t') points taken up by Bob's ruler. This will sweep out an area in the spacetime diagram. The length of Bob's ruler in its rest frame is 2 meters, say. The following diagram shows how the ruler

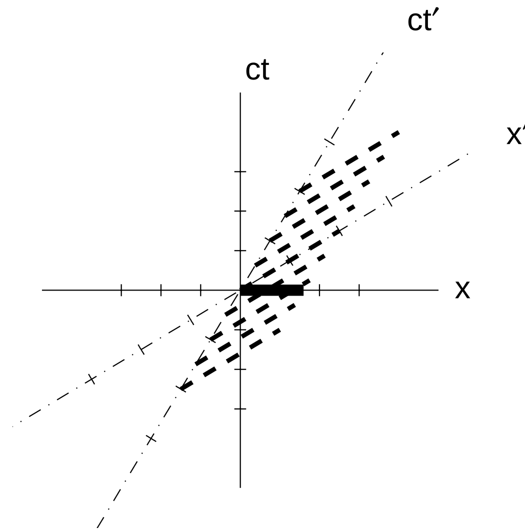
- Moves forward in time in both Bob's and Bill's frame
- Moves forward in position only in Bill's frame (ie. is at rest in Bob's frame)



If Rocket Bob wants to make a length measurement, it would only make sense to make a measurement of each end of the object at the same time. It wouldn't make sense, for instance, to measure the position of one end of the object, wait a few seconds, then measure the other end because the object could have moved in that time. One wants to do length measurements by the *simultaneous* measurement of the position of each end of an object. In a space-time diagram *simultaneous* implies *parallel* to the x (or x') axis, depending on who is doing the measurement. Below shows the measurement of the length of Bob's ruler by both Bob and Bill.

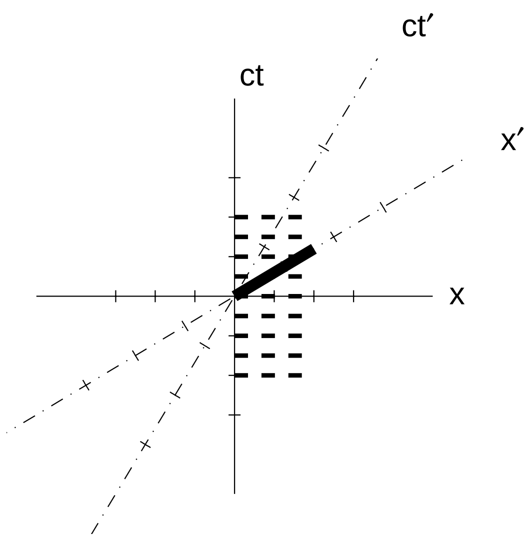


Bob's ruler measured by Bob

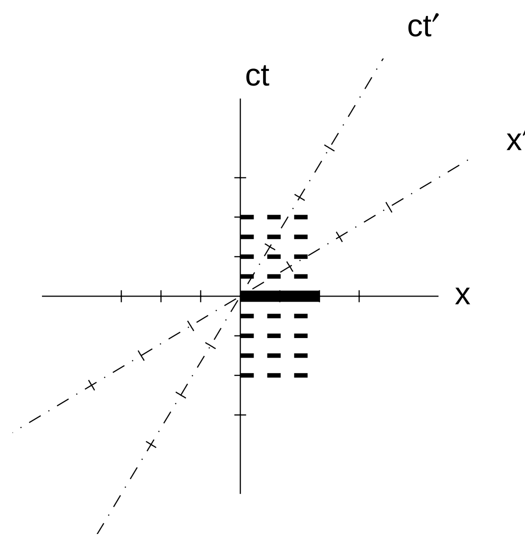


Bob's ruler measured by Bill

The diagram demonstrates how Bill can measure Bob's ruler as shorter than 2 meters. According to Bob, however, Bill has not measured each end simultaneously, but has in fact measured the farther end a little bit late, after Bob as moved a little bit farther away. They don't agree on simultaneity so they cannot agree on lengths. Likewise, the following diagram shows how Bill's ruler can be seen as shorter than 2 meters when Bob measures it.



Bill's ruler measured by Bob



Bill's ruler measured by Bill

The diagrams are a convenient way of pointing out the fact there is truly no contradiction the claims of each observer. In order to figure out just how much of a change is perceived, one needs to go to the math.

The Math

Bob's Ruler

The first step is to write down all that is known about the events in which we are interested, namely each end of Bob's ruler. This will give us two points in space-time, (x_1, ct_1) and (x_2, ct_2) (or, if you're Bob, (x'_1, ct'_1) and (x'_2, ct'_2)). For simplicity we will follow the standard set in the diagrams, that the origin is at the left end of the ruler. Thus, all the known quantities are

$$\begin{aligned}
\text{(velocity): } v &= \frac{3}{5}c \\
\beta \equiv \frac{v}{c} &= \frac{3}{5} \\
\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} &= \frac{5}{4} \\
\text{(origin): } x_1 = x'_1 = ct'_1 = ct_1 &= 0 \\
\text{(length of ruler measured by Bob): } x'_2 - x'_1 &= 2m \\
\text{(Bill measures each end simultaneously): } ct_2 - ct_1 &= 0
\end{aligned}$$

The question we ask is *what length does Bill measure Bob's ruler?*.

$$\begin{aligned}
\text{(from Equation 2.5): } (x'_2 - x'_1) &= \gamma(x_2 - x_1) - \beta\gamma(ct_2 - ct_1) \\
2m &= \left(\frac{5}{4}\right) \cdot (x_2 - 0) - \left(\frac{3}{5}\right) \cdot \left(\frac{5}{4}\right) \cdot (0 - 0) \\
x_2 &= \frac{4}{5}2m = 1\frac{3}{5}m
\end{aligned}$$

which is shorter than $2m$.

The one last quantity that we don't know is t'_2 or *when, according to Bob, did Bill measure the second end of the ruler?*

$$\begin{aligned}
\text{(from Equation 2.6): } (ct'_2 - ct'_1) &= \gamma(ct_2 - ct_1) - \beta\gamma(x_2 - x_1) \\
ct'_2 &= -\left(\frac{3}{5}\right) \cdot \left(\frac{5}{4}\right) \cdot (1\frac{3}{5}m) \\
&= -1\frac{1}{5}m
\end{aligned}$$

which is *early*! Bob and Bill do not agree on simultaneity.

Bill's Ruler

Again we write down all that is known, and then use the Lorentz transformation equations to get all the rest.

$$\begin{aligned}
\text{(velocity): } v &= \frac{3}{5}c \\
\beta &= \frac{3}{5}
\end{aligned}$$

$$\begin{aligned}
\gamma &= \frac{5}{4} \\
\text{(origin): } x_1 = x'_1 = ct'_1 = ct_1 &= 0 \\
\text{(length of ruler measured by Bill): } x_2 - x_1 &= 2m \\
\text{(Bob measures each end simultaneously): } ct'_2 - ct'_1 &= 0
\end{aligned}$$

What length does Bob measure Bill's ruler?.

$$\begin{aligned}
\text{(from Equation 2.3): } (x_2 - x_1) &= \gamma(x'_2 - x'_1) + \beta\gamma(ct'_2 - ct'_1) \\
2m &= \left(\frac{5}{4}\right) \cdot (x'_2 - 0) \\
x'_2 &= \frac{4}{5} 2m = 1\frac{3}{5}m
\end{aligned}$$

which is same length Bill that measured Bob's ruler to be.

When did Bob measure the second end of Bill's ruler?

$$\begin{aligned}
\text{(from Equation 2.4): } (ct_2 - ct_1) &= \gamma(ct'_2 - ct'_1) + \beta\gamma(x'_2 - x'_1) \\
ct_2 &= \left(\frac{3}{5}\right) \cdot \left(\frac{5}{4}\right) \cdot (1\frac{3}{5}m) \\
&= 1\frac{1}{5}m
\end{aligned}$$

which is *late*! This makes sense when you think about which direction Bob sees Bill's ruler moving.

Summary

If we just look at the lengths of the rulers, we can say that a moving ruler is measured as contracted by a stationary observer. The amount of contraction is given by the factor γ (which is always greater than one). Thus

$$(\text{length})_{\text{contracted}} = \frac{1}{\gamma} \times (\text{length})_{\text{uncontracted}}$$

Note that $(\text{length})_{\text{contracted}}$ is measured simultaneously in a frame where the ruler is moving, and $(\text{length})_{\text{uncontracted}}$ is measured simultaneously in a frame where the ruler is *not* moving. The two measurements do *not* agree on what is simultaneous, so taking length contraction alone leads to apparent "paradoxes".

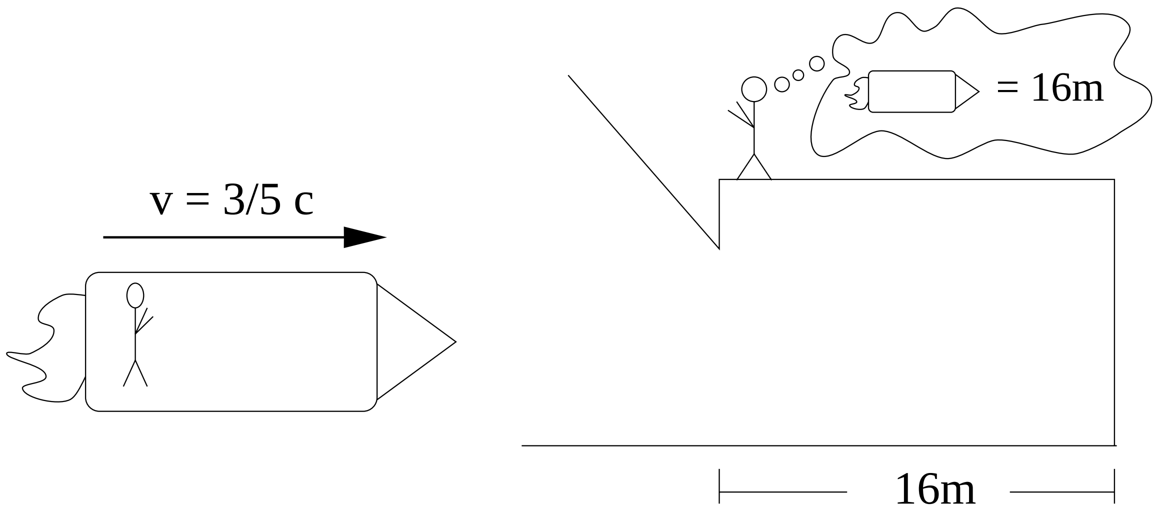
A similar relation holds for time dilation.

$$(\text{time})_{\text{dilated}} = \gamma \times (\text{time})_{\text{undilated}}$$

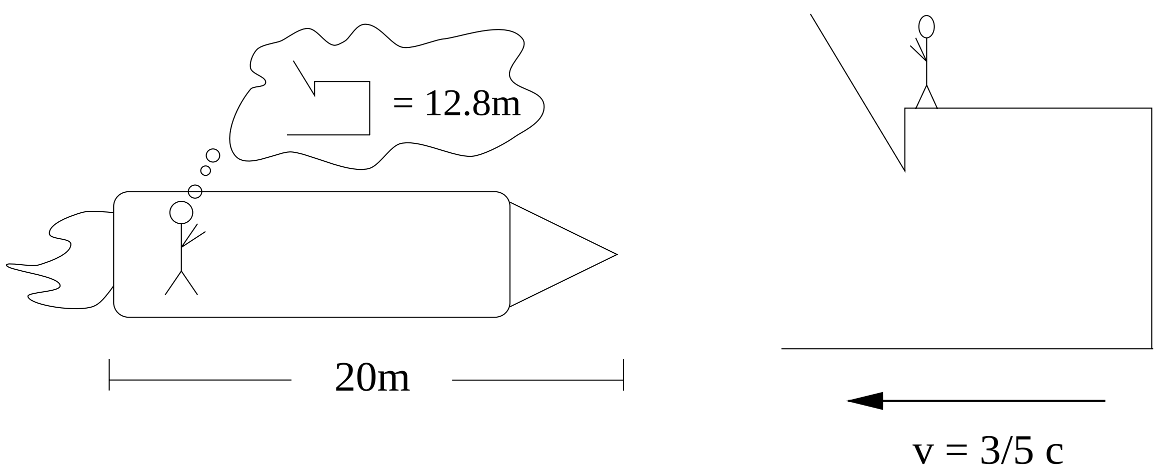
Of course, similar "paradoxes" occur in this case, when taken out of the context of simultaneity.

6 Example 3: Barn and Pole "Paradox"

The so-called Barn and Pole "paradox" is the following. Cowboy Bill realizes that Rocket Bob's ruler is reduced in length by 5/4, and that this length contraction is not limited merely to the ruler in the rocket but includes the rocket itself. Therefore the $20m$ rocket appears to be only $16m$. “Ah ha!,” thinks Bill. “I just so happen to have a $16m$ barn into which I can fit the rocket!. Of course I will have fortify the barn to be able to take the impact of the rocket, but after that, all I need to do is close the barn door at the instant the rocket strikes the inside wall.” The following picture illustrates Cowboy Bill’s sinister plan.



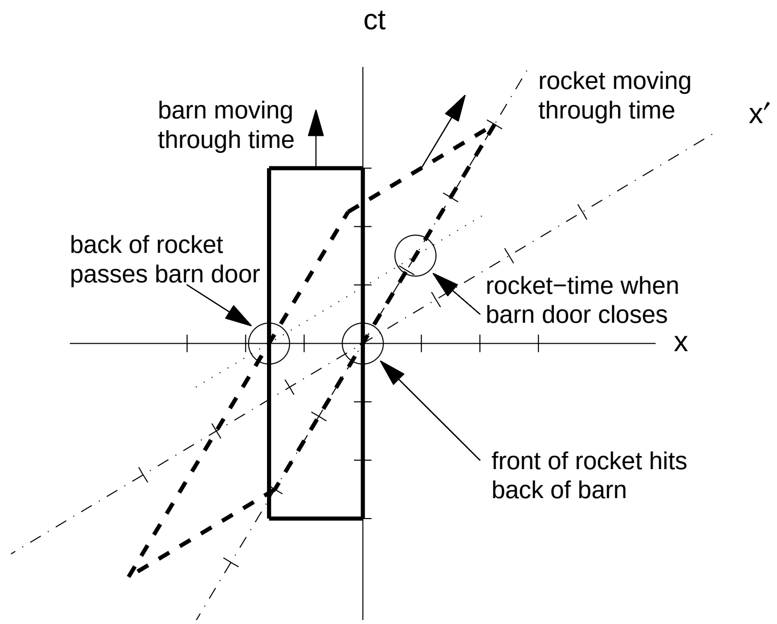
Rocket Bob realizes what Cowboy Bill is up to, but is not worried. Looking out the window in the rocket, Bob sees Bill and Bill’s barn moving toward him at $v = 3/5c$, so all of Bill’s lengths are contracted by 5/4. “My 20m rocket will never fit into that barn!” exclaims Bob. “ It’s only 12.8 meters long! When I hit the inside of the barn, there will be 7.2 meters of rocket sticking out the back, making it impossible for Cowboy Bill to close the door!” Bob’s plan is illustrated in the following picture.



Which one is right? Can they *both* be right, as in the previous example? No! Here’s why. Despite differences in coordinate systems, any prediction of a physical event *must* be the same for both frames. The rocket sticking out of the back either happens or it doesn’t. The *reasons* each person gives for the physical event may differ, but the outcome has to be identical.

The Picture

The picture is almost exactly the same as the previous example. If we look at the path the barn sweeps through space-time, and the path that the rocket sweeps through space-time, we can immediately see the answer.



In the picture we see the barn sweeping through space-time, as it sits there in the unprimed frame (the x position doesn't change, but time advances). The rocket also is stationary, but in the primed frame (it sweeps out a path parallel to the t' axis). From the point of view of Cowboy Bill, in the unprimed frame, the rocket moves to the right (as t increases, so does x). One can clearly see that at $t = 0$, Cowboy Bill measures the rocket to be inside the barn. Of course, the two points he measures ($x = 0, ct = 0$ and $x = -16m, ct = 0$) are *not* at the same time in Rocket Bob's frame. The first point is at $ct' = 0$ and the second point is at some time later $ct' > 0$. In that amount of time, the back of the rocket moves to be just inside the barn. The mathematics bears this out.

The Math

The answer can also be seen quite clearly by writing down the coordinates of each of the events in question, as we did with the rulers.

The events we have are the position and time for the rocket hitting the back of barn (all the way inside) and the position and time for the door at the front of the barn to close. Again the unprimed frame (x and ct) are Cowboy Bill on the barn, and the primed frame (x' and ct') is for Rocket Bob.

All of the known quantities are

$$\begin{aligned}
\text{(velocity): } v &= \frac{3}{5}c \\
\beta \equiv \frac{v}{c} &= \frac{3}{5} \\
\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} &= \frac{5}{4} \\
\text{(rocket hits the back of the barn): } x_1 = x_1' = ct_1' = ct_1 &= 0 \\
\text{(length of rocket measured by Bob):} &= 20m \\
\text{(length of barn measured by Bill):} &= 16m
\end{aligned}$$

We have already figured out the contracted length of the barn as measured from the rocket frame, and the contracted length of the rocket as measured from the barn.

$$\begin{aligned}\text{(length of rocket measured by Bill):} \quad &= \frac{1}{\gamma} \times 20m = 16m \\ \text{(length of barn measured by Bill):} \quad &= \frac{1}{\gamma} \times 16m = 12.8m\end{aligned}$$

Now we need to write down the coordinates of each of the events, according to Bill and Bob

The Story According to Bill

$$\begin{aligned}\text{(position where rocket hits the back of the barn):}\ x_1 &= 0 \\ \text{(time when rocket hits the back of the barn):}\ ct_1 &= 0 \\ \text{(position where barn door closes):}\ x_2 &= -16m\end{aligned}$$

Because the rocket is $16m$ long, as measured by Bill, he closes the barn door at the same time that the rocket hits the front.

$$\text{(time when barn door closes):}\ ct_2 = 0$$

So, according to Bill, he captures the rocket.

The Story According to Bob

$$\begin{aligned}\text{(position where rocket hits the back of the barn):}\ x'_1 &= 0 \\ \text{(time when rocket hits the back of the barn):}\ ct'_1 &= 0\end{aligned}$$

Using Equations 2.5 and 2.6, and the coordinates given by Bill's story, we can figure out where and when (according to Bob) the barn door closes.

$$\begin{aligned}\text{(position where barn door closes):}\ x'_2 &= \gamma x_2 - \beta\gamma(ct_2) \\ &= \left(\frac{5}{4}\right) \cdot (-16m) - \left(\frac{3}{5}\right) \cdot \left(\frac{5}{4}\right) \cdot (0) = -20m\end{aligned}$$

which means that the door closes right on the back of the rocket, without chopping it off! How can this be, if the barn (according to Rocket Bob) is only $12.8m$ long? Let's look at the *time* that this occurs.

$$\begin{aligned}\text{(time when barn door closes):}\ ct'_2 &= \gamma(ct_2) - \beta\gamma x_2 \\ &= \left(\frac{5}{4}\right) \cdot (0) - \left(\frac{3}{5}\right) \cdot \left(\frac{5}{4}\right) \cdot (-16m) = +12m\end{aligned}$$

which is late! According to Rocket Bob, the barn door is not closed right when the front of the rocket strikes the inside wall of the barn, but is delayed. How long is this delay, anyway? Let's look at how far the barn would travel in that time (remember, this is from Rocket Bob's perspective).

$$\begin{aligned}\text{(distance traveled by barn in } +12m \text{ of time): } \beta \times ct'_2 &= \left(\frac{3}{5}\right) \cdot (12m) \\ &= 7.2m\end{aligned}$$

which is exactly the distance that the rocket would have stuck out, had the door been closed at the instant (according to Rocket Bob) that his rocket struck the inside wall of the barn!

Summary

The answer is that the rocket does get stuck in the barn, but the interpretations of each party involved is different. Cowboy Bill, on the barn, says that the rocket is contracted and fits neatly into the barn. Rocket Bob says that Bill didn't close the barn door at the right time, but waited until the rear of the rocket made it inside the barn. The second interpretation has the peculiar property that the back of the rocket is not aware that the front of the rocket has stopped! Of course, a light signal (the fastest method of transferring information) would not reach the back of the rocket in time to prevent this.